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Continuous Mixture of Copulas: A Bayesian Approach

Ralph dos Santos Silva[a]

[a] *Instituto de Matemática, Universidade Federal de Rio de Janeiro, Brazil*

ABSTRACT
This paper aims to demonstrate how new copulas can be derived through mixtures of copulas and their associated marginal distributions. We investigate the properties of these copulas by applying various dependency measures and conducting numerical evaluations via Monte Carlo integration. The study considers six bivariate copula families and asymmetric marginal distributions. To illustrate the methodology, we provide a simple example using the Clayton copula with exponential marginals, as well as the product copula with gamma marginals. In the applications, one of the six copulas is combined with the product copula, where the mixture is performed on the scale of the asymmetric normal distribution with an inverse gamma distribution. This results in an asymmetric Student-t distribution, generating a new copula. We estimate the model parameters using a Bayesian approach, with model selection based on the deviance information criterion (DIC) and similar metrics. For real-world applications, we apply the models to a dataset of insurance claims and two sets of daily returns from European and Latin American markets.

KEYWORDS
Dependency Measure, Markov Chain Monte Carlo, Model Selection, Scale Mixture

1. Introduction

There have been several applications of multivariate distributions constructed by combining univariate marginal distributions with copula functions [9, 11, 13]. Copula functions are multivariate distributions defined on the unit hypercube $[0,1]^d$, where each univariate marginal distribution is uniform on the interval $(0,1)$. Specifically, the copula function C is defined as

$$C(u_1,\ldots,u_d) = \Pr(U_1 \leqslant u_1,\ldots,U_d \leqslant u_d) \tag{1}$$

where $U_i \sim \text{Uniform}(0,1)$ for $i = 1,2,\ldots,d$. Furthermore, if $(X_1,\ldots,X_d) \in \mathbb{R}^d$ has a continuous multivariate distribution with the joint cumulative distribution function $H(x_1,\ldots,x_d) = \Pr(X_1 \leqslant x_1,\ldots,X_d \leqslant x_d)$, then by Sklar's theorem [22], there exists a unique copula function $C : [0,1]^d \to [0,1]$ corresponding to H such that

$$H(x_1,\ldots,x_d) = C(H_1(x_1),\ldots,H_d(x_d))$$

CONTACT Author[a]. Email: ralph@im.ufrj.br

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for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and continuous marginal distributions $H_i(x_i) = \Pr(X_i \leqslant x_i)$, where $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots, d$. Additionally, if C is a copula on $[0,1]^d$ and $H_1(x_1), \ldots, H_d(x_d)$ are cumulative distribution functions on $\mathbb{R}$, then the joint cumulative distribution function

$$H(x_1, \ldots, x_d) = C(H_1(x_1), \ldots, H_d(x_d))$$

is a valid cumulative distribution function on $\mathbb{R}^d$ with univariate marginal distributions $H_1(x_1), \ldots, H_d(x_d)$. For a thorough theoretical discussion on copulas, refer to Joe [15] and Nelsen [18].

Flexible multivariate distributions can be formulated using predetermined discrete and/or continuous marginal distributions, along with a copula function that captures the intended dependence structure. Silva and Lopes [21] discuss one- and two-step estimation techniques for copulas, highlighting the significance of employing a fully Bayesian one-step approach. This paper aligns with their perspective by utilizing continuous mixtures of copulas and implementing a full Bayesian framework through Markov chain Monte Carlo methods [8]. Moreover, we use the deviance information criterion (DIC) [23] and similar criteria to choose the most suitable continuous mixture copula-based model.

This paper presents a novel method for constructing copulas through continuous mixtures of copulas and their marginal distributions, building upon existing work in the literature. We analyze the properties of the resulting copulas using various dependency measures. The concept of mixture results based on conditional distributions and Laplace transforms was first introduced by Joe [15], and Nelsen [18] expanded on mixtures in copula parameters. To illustrate the approach, we provide a simple example featuring the Clayton copula with exponential marginals and the product copula with gamma marginals. For both synthetic and real-world applications, we combine six bivariate copulas with the product copula, incorporating asymmetric normal and inverse gamma distributions. This results in an asymmetric Student-t marginal distribution and a new copula. The method is demonstrated using actuarial and financial data.

The rest of the paper is organized as follows. In Section 2, we outline the mixture of copulas through marginal distributions and then address how to compute the dependency measures. Section 3 discusses inference in the posterior distribution via Markov chain Monte Carlo methods and model selection using the deviance information criterion. Section 4 presents three real-world applications, one related to insurance claims and the other two focused on daily return indices. Our concluding remarks are provided in Section 5.

2. Mixture of copulas

In this section, we introduce the concept of copula mixtures, discuss how dependency measures can be represented, and offer suggestions for evaluating them numerically using Monte Carlo integration. To begin, we present the most commonly used copula functions (see Equation (1)) in Table 1.

The parameter $\theta \in [-1, 1]$ applies to the Gaussian and Student-t copula functions. Additional discussions and properties of these copula functions can be found in [2], [5], [10], and [12] (for Gumbel and Gaussian), and [3] (for Student-t).

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be random vectors. F will represent the cumulative distribution

Table 1. Six bivariate copula functions.

Cópula	$C(u,v\|\theta)$	$\theta \in \Omega$
Clayton	$(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$	$(0,\infty)$
Frank	$-\frac{1}{\theta}\log\left(1+\frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{\exp(-\theta)-1}\right)$	$\mathbb{R}\backslash\{0\}$
Gaussian	$\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-\theta^2}}\exp\left\{\frac{2\theta st-s^2-t^2}{2(1-\theta^2)}\right\}dsdt$	$[-1,1]$
Gumbel	$\exp\left\{-[(-\log u)^\theta+(-\log v)^\theta]^{1/\theta}\right\}$	$[1,\infty)$
Heavy tail	$u+v-1+[(1-u)^{-1/\theta}+(1-v)^{-1/\theta}-1]^{-\theta}$	$(0,\infty)$
Student-t	$\int_{-\infty}^{T_\nu^{-1}(u)}\int_{-\infty}^{T_\nu^{-1}(v)}\frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\nu\pi\sqrt{1-\theta^2}}\left(1+\frac{s^2+t^2-2\theta st}{\nu(1-\theta^2)}\right)dsdt$	$[-1,1]$

function (cdf) of $\boldsymbol{X}$ given $\boldsymbol{Y}$, and its univariate marginals will be denoted by F_i, for $i=1,\dots,d$. G and H will represent the cdf of $\boldsymbol{Y}$ and $\boldsymbol{X}$, respectively, with their univariate marginals denoted by G_j, for $j=1,\dots,m$, and H_i, for $i=1,\dots,d$.

Theorem 2.1. *Let $\boldsymbol{X}=(X_1,\dots,X_d)$ and $\boldsymbol{Y}=(Y_1,\dots,Y_m)$ be random vectors such that $\boldsymbol{X}|\boldsymbol{Y}$ has a cdf given by $C_{\boldsymbol{X}|\boldsymbol{Y}}(F_1(x_1|\boldsymbol{y}),\dots,F_d(x_d|\boldsymbol{y}))$, and $\boldsymbol{Y}$ has a cdf given by $C_{\boldsymbol{Y}}(G_1(y_1),\dots,G_m(y_m))$. Then, $\boldsymbol{X}$ has a cdf given by*

$$C_{\boldsymbol{X}}(H_1(x_1),\dots,H_d(x_d))=\int C_{\boldsymbol{X}|\boldsymbol{Y}}(F_1(x_1|\boldsymbol{y}),\dots,F_d(x_d|\boldsymbol{y}))dC_{\boldsymbol{Y}}(G_1(y_1),\dots,G_m(y_m))$$

where $\boldsymbol{y}=(y_1,\dots,y_m)\in\mathbb{R}^m$. Furthermore, $C_{\boldsymbol{X}}$ is a copula function.

Proof: This follows from Sklar's theorem and the fact that $H(\boldsymbol{x})=\int F(\boldsymbol{x}|\boldsymbol{y})dG(\boldsymbol{y})$, where $\boldsymbol{x}=(x_1,\dots,x_d)\in\mathbb{R}^d$. To verify that $C_{\boldsymbol{X}}$ is indeed a copula function, we show the following two properties:

(1) $\lim_{u_i\to 0} C_{\boldsymbol{X}}(u_1,\dots,u_i,\dots,u_d)=0$; and
(2) $C_{\boldsymbol{X}}(1,\dots,1,u_i,1,\dots,1)=u_i$ for $i=1,\dots,d$.

These properties hold using the fact that $C_{\boldsymbol{X}|\boldsymbol{Y}}$ is a copula. The other properties follow similarly.

Theorem 2.1 states that a mixture of copulas results in another copula. It is also possible to extend this result by mixing the parameters of the copula [18], $C_{\boldsymbol{X}}$, or equivalently in $C_{\boldsymbol{X}|\boldsymbol{Y}}$ and $C_{\boldsymbol{Y}}$. The result of Theorem 2.1 includes both discrete and/or continuous mixtures. Moreover, $C_{\boldsymbol{X}|\boldsymbol{Y}}$ and $C_{\boldsymbol{Y}}$ can be any copulas. They determine the structure of asymmetry and dependence in the resulting copula $C_{\boldsymbol{X}}$, while the choice of marginal distributions in the mixture determines the characteristics and properties of the marginal distributions of $\boldsymbol{X}$. A hierarchical copula model can be an interesting application of the copula mixture approach.

Theorem 2.2. *Any lower-order marginal distribution ($k<d$) of $C_{\boldsymbol{X}}$ can be obtained as a mixture of the marginal $C_{\boldsymbol{X}|\boldsymbol{Y}}$ of the same dimension and the copula $C_{\boldsymbol{Y}}$, that is,*

$$C_{\boldsymbol{X}}(H_{s_1}(x_{s_1}),\dots,H_{s_t}(x_{s_t}))=\int C_{\boldsymbol{X}|\boldsymbol{Y}}(F_{s_1}(x_{s_1}|\boldsymbol{y}),\dots,F_{s_t}(x_{s_t}|\boldsymbol{y}))dC_{\boldsymbol{Y}}(G_1(y_1),\dots,G_m(y_m))$$

where $s_1,\dots,s_t\in\{1,\dots,d\}$.

Proof: Let the limits $x_\ell \to \infty$ for all $\ell \neq s_1, \ldots, s_t$. We then interchange the order of limits and integrals since they are all well-defined. Next, we apply the marginalization property of a copula, $C_{\boldsymbol{X}|\boldsymbol{Y}}$, to obtain the result.

Remark 2.1. *If $d = m$ and X_i only depends on Y_i, which in turn is independent of all other X_ℓ (for $\ell \neq i$), then*

$$H_i(x_i) = \int F_i(x_i|y_i)dG_i(y_i).$$

Remark 2.1 is significant because it helps determine the resulting marginal distributions in most applications. For instance, suppose X_i is a normal random variable with mean 0 and variance Y_i, and Y_i follows an inverse gamma distribution with parameters $\nu/2$ and $\nu/2$. In this case, the resulting marginal distribution will follow a Student's t-distribution with ν degrees of freedom, regardless of the copulas used in the mixture. This outcome is independent of the copulas $C_{\boldsymbol{X}|\boldsymbol{Y}}$ and $C_{\boldsymbol{Y}}$, but it can provide insights into certain characteristics of the resulting copula $C_{\boldsymbol{X}}$. The results outline the general structure of the mixture, its cumulative distribution function, and marginals, whilst key features of the copulas are captured through dependence measures.

For copulas, the well-known Kendall's τ measure [16, 18] for the resulting copula $C_{\boldsymbol{X}}$ is given by:

$$\tau_{\boldsymbol{X}} = 4\int C_{\boldsymbol{X}}(H_{s_1}(x_{s_1}), H_{s_2}(x_{s_2}))dC_{\boldsymbol{X}}(H_{s_1}(x_{s_1}), H_{s_2}(x_{s_2})) - 1,$$

where $s_1, s_2 \in \{1, \ldots, d\}$, and $C_{\boldsymbol{X}}(H_{s_1}(x_{s_1}), H_{s_2}(x_{s_2}))$ is given by Theorems 2.1 and 2.2. In most cases, $\tau_{\boldsymbol{X}}$ does not have a closed-form expression but can be evaluated numerically, for example, using Monte Carlo integration as

$$\tau_{\boldsymbol{X}} \simeq \frac{4}{LK}\left[\sum_{\ell=1}^{L}\sum_{k=1}^{K} C_{\boldsymbol{X}|\boldsymbol{Y}}(F_{s_1}(x_{s_1}^{(\ell)}|y^{(k)}), F_{s_2}(x_{s_2}^{(\ell)}|y^{(k)}))\right] - 1. \tag{2}$$

where $\boldsymbol{y}^{(\ell)}$ is draw from $C_{\boldsymbol{Y}}(G_1(y_1), \ldots, G_m(y_m))$, and $(x_{s_1}^{(\ell)}, x_{s_2}^{(\ell)})$ is a draw from $C_{\boldsymbol{X}}(H_{s_1}(x_{s_1}), H_{s_2}(x_{s_2}))$. We use this measure in our applications.

The Spearman's ρ measure [16, 18] for the resulting copula $C_{\boldsymbol{X}}$ is given by

$$\rho_{\boldsymbol{X}} = 12\int H_{s_1}(x_{s_1})H_{s_2}(x_{s_2})C_{\boldsymbol{X}}(H_{s_1}(x_{s_1}), H_{s_2}(x_{s_2})) - 3$$

where

$$H_{s_t}(x_{s_t}) = \int F_{s_t}(x_{s_t}|y)dC_{\boldsymbol{Y}}(G_1(y_1), \ldots, G_m(y_m)), \quad \text{for} \quad t = 1, 2,$$

and $s_1, s_2 \in \{1, \ldots, d\}$. Again, we can approximate $\rho_{\boldsymbol{X}}$ using Monte Carlo integration as

$$\rho_{\boldsymbol{X}} \simeq \frac{12}{L}\left[\sum_{\ell=1}^{L} H_{s_1}(x_{s_1}^{(\ell)})H_{s_2}(x_{s_2}^{(\ell)})\right] - 3 \tag{3}$$

if H_{s_1} and H_{s_2} are known in closed form. Otherwise, we have

$$\rho_{\boldsymbol{X}} \simeq \frac{12}{LK^2}\left\{\sum_{\ell=1}^{L}\left[\sum_{k=1}^{K} F_{s_1}(x_{s_1}^{(\ell)}|\boldsymbol{y}^{(k)})\right]\left[\sum_{k=1}^{K} F_{s_2}(x_{s_2}^{(\ell)}|\boldsymbol{y}^{(k)})\right]\right\} - 3. \tag{4}$$

For both Kendall's τ and Spearman's ρ, simple versions of the approximations can be computed using their sample counterparts.

The lower and upper tail indices [18] of the resulting copula $C_{\boldsymbol{X}}$ are given respectively by

$$\lambda_L = \lim_{z\to-\infty}\frac{r(z)}{H_{s_t}(z)} \quad \text{and} \quad \lambda_U = \lim_{z\to\infty}\frac{1}{1-H_{s_t}(z)}\left[1 - 2H_{s_t}(z) - r(z)\right],$$

where

$$r(z) = \int C_{\boldsymbol{X}|\boldsymbol{Y}}(F_{s_1}(z|\boldsymbol{y}), F_{s_2}(z|\boldsymbol{y}))dC_{\boldsymbol{Y}}(G_1(y_1),\ldots,G_p(y_p)).$$

An approximation for the value of λ_L can be obtained through a local regression using least squares methods and taking the intercept as the approximate value. That is, let $w_r = \alpha + \xi_1 z_r + \xi_2 z_r^2$ be a local linear regression with z_r being a decreasing sequence and

$$w_r \simeq \left[\sum_{k=1}^{K} F_{s_t}(z_r|\boldsymbol{y}^{(k)})\right]^{-1}\left[C_{\boldsymbol{X}|\boldsymbol{Y}}(F_{s_1}(z_r|\boldsymbol{y}^{(k)}), F_{s_2}(z_r|\boldsymbol{y}^{(j)}))\right].$$

Then $\lambda_L \simeq \hat{a}$, where $\hat{a}$ is the least squares estimate from the above regression. The same idea can be applied to the upper tail index λ_U.

The measure $D(q)$, consequently the Schweizer and Wolff measure σ [20], and the dependence index Φ^2, for the resulting copula $C_{\boldsymbol{X}}$ are given by

$$D_{\boldsymbol{X}}(q) = \left[\kappa_q \int\int |C_{\boldsymbol{X}}(H_{s_1}(x_{s_1}), H_{s_2}(x_{s_2})) - H_{s_1}(x_{s_1})H_{s_2}(x_{s_2})|^q dH_{s_1}(x_{s_1})dH_{s_2}(x_{s_2})\right]^{1/q}$$

where $\sigma_{\boldsymbol{X}} = D_{\boldsymbol{X}}(1)$ with $\kappa_1 = 12$ and $\Phi^2_{\boldsymbol{X}} = (D_{\boldsymbol{X}}(2))^2$ with $\kappa_2 = 90$. A numerical approximation of $D_{\boldsymbol{X}}(q)$ by Monte Carlo integration is given by

$$\begin{aligned} D_{\boldsymbol{X}}(q) \simeq \Bigg\{\frac{\kappa_q}{L}\sum_{\ell=1}^{L}\Bigg|&\left[\frac{1}{K}\sum_{k=1}^{K} C_{\boldsymbol{X}|\boldsymbol{Y}}(F_{s_1}(x_{s_1}^{(\ell)}|\boldsymbol{y}^{(k)}), F_{s_2}(x_{s_2}^{(\ell)}|\boldsymbol{y}^{(k)}))\right] \\ &-\left[\frac{1}{J}\sum_{j=1}^{J} F_{s_1}(x_{s_1}^{(\ell)}|\boldsymbol{y}^{(j)})\right]\left[\frac{1}{M}\sum_{m=1}^{M} F_{s_2}(x_{s_2}^{(\ell)}|\boldsymbol{y}^{(m)})\right]\Bigg|^q\Bigg\}^{1/q}. \end{aligned} \tag{5}$$

In Example 2.1, we want to illustrate the idea of mixture in copulas, how we can obtain marginal distributions, and what the results of all these dependence measures are.

Example 2.1. *Let X_i and Y_i, $i = 1, 2$, be random variables such that $(X_i|Y_i) \sim$ Exponential(y_i) with mean $E(X_i|Y_i = y_i) = 1/y_i$ and $Y_i \sim$ Gamma(λ_i, η_i) with mean $E(Y_i) = \lambda_i/\eta_i$ and variance Var$(Y_i) = \lambda_i/\eta_i^2$. It is easy to prove that X_i has pdf $h_i(x_i) = \lambda_i\eta_i^{\lambda_i}/(\eta_i+x_i)^{\lambda_i+1}$ and cdf $H_i(x_i) = 1-\eta_i^{\lambda_i}/(\eta_i+x_i)^{\lambda_i}$. Furthermore, consider $C_{\boldsymbol{X}|\boldsymbol{Y}}$ as the Clayton copula given in Table 1 whose pdf can be easily computed [18] (see also Table A1 in Appendix A) and $C_{\boldsymbol{Y}}$ the product copula $C^{\perp}$. Setting $\lambda_1 = \lambda_2 = 2$, $\eta_1 = \eta_2 = 2$, $\theta = 2$, and using the sampling importance resampling algorithm [8] with a resampling rate of 5% to generate values of $\boldsymbol{X}|\boldsymbol{Y}$, we get:*

(1) *$\tau^S_{\boldsymbol{X}} \simeq 0.328$ and $\tau^\star_{\boldsymbol{X}} \simeq 0.334$, where S corresponds to the sample version of Kendall's τ measure for a sample size of 5,000 and $\star$ the approximation in Equation (2) with $L = 10,000$ and $K = 50,000$. Note that $\tau^S_{\boldsymbol{X}}$ gives the same result with a smaller sample size. The Kendall τ measure for the Clayton copula without mixture is 0.50, with the sample generated using the mean of y_i, $E(Y_i) = 1$ for $i = 1, 2$.*

(2) *$\rho^S_{\boldsymbol{X}} \simeq 0.473$, $\rho^\star_{\boldsymbol{X}} \simeq 0.482$ and $\rho^\dagger_{\boldsymbol{X}} \simeq 0.477$, where S corresponds to the sample version of Spearman's ρ measure for a sample size of 5,000, $\star$ the approximation in Equation (3) with $L = 10,000$, and $\dagger$ the approximation in Equation (4) with $L = 10,000$ and $K = 50,000$. Note that $\rho^S_{\boldsymbol{X}}$ gives the same result with a smaller sample size. Spearman's ρ measure for the Clayton copula without mixture is 0.68.*

(3) *$\lambda_{\boldsymbol{X}} \simeq 0.532$ using the local regression model idea with the sequence z_r going to 0 and y_r. The lower tail index for the Clayton copula without mixture is approximately 0.707.*

(4) *$\sigma_{\boldsymbol{X}} \simeq 0.474$ and $\Phi^2 \simeq 0.205$ using Equation (5) with $L = 10,000$ and $J = K = M = 50,000$. The Schweizer and Wolff measure σ and the dependence index Φ for the Clayton copula without mixture are approximately 0.68 and 0.43, respectively.*

Now, it can be seen that all the measures indicate that the dependence between the two variables X_1 and X_2 decreases when compared to the dependence measures of the Clayton copula without mixture. A plausible explanation for this is the choice of $C_{\boldsymbol{Y}}$ as the product copula, that is, independent marginals. Another possibility for the mixture would be using the Clayton copula in both $C_{\boldsymbol{X}|\boldsymbol{Y}}$ and $C_{\boldsymbol{Y}}$. It is important to note that the numerical calculation of the dependency measures mentioned above can be computationally intensive. Special attention should be given to the sample versions of Kendall's τ and and Spearman's ρ, which are used in our applications.

3. Inference in the posterior distribution

We consider the Bayesian approach, where all unknown quantities are estimated from the posterior distribution. The exception is for the Student copula, where the degrees of freedom are given and fixed. Formally, $\pi(\Psi) \propto L(\Psi)p(\Psi)$, where Ψ is a vector of parameters (all unknown quantities), including those of the copula and the marginal distributions, while $\pi(\Psi)$, $L(\Psi)$, and $p(\Psi)$ are the posterior distribution, the likelihood function, and the prior distribution, respectively. To facilitate understanding, we focus on bivariate continuous distributions and copulas. Let (X_1, X_2) be a continuous bivariate random variable with copula $C_{\boldsymbol{X}}$. Thus, its joint pdf is given by:

$$h(x_1, x_2|\Psi) = c_{\boldsymbol{X}}(H_1(x_1|\Psi), H_2(x_2|\Psi)|\theta)h_1(x_1|\Psi)h_2(y_2|\Psi), \tag{6}$$

where h_i and H_i represent the pdf and cdf of the marginal distribution, respectively. The copula density, $c_{\boldsymbol{X}}$, is the pdf calculated as the mixed derivative of $C_{\boldsymbol{X}}$, if it exists, with respect to x_1 and x_2. For example, see in Table A1 in the Appendix A the pdf of the six families of copulas presented in Table 1.

Let $((x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n}))$ be a sample of size $n \geqslant 1$ of i.i.d. data from the bivariate joint distribution with pdf given by Equation (6). Then,

$$L(\Psi) = \prod_{i=1}^{n} c_{\boldsymbol{X}}(H_1(x_{1i}|\Psi), H_2(x_{2i}|\Psi)|\theta) h_1(x_{1i}|\Psi) h_2(x_{2i}|\Psi).$$

The model specification is completed by assigning a prior distribution $p(\Psi)$ to the parameter vector Ψ. All parameters are assumed to be independent a priori, each with its own relatively vague prior distribution. In the case of the mixture model, $L(\Psi)$ is composed by the $\boldsymbol{X}|\boldsymbol{Y}$ and $\boldsymbol{Y}$ distributions, with parameters ommited from the notation.

Since copula functions generally do not have simple forms, the posterior distribution is generally not available in a closed form. Inference in the posterior distribution is facilitated by Markov chains Monte Carlo methods. Specifically, the Gibbs sampler was chosen for the applications, and for most of the complete conditionals, we use the slice sampler [17], except when some of the complete conditionals have a known and easy-to-sample closed form. These choices induce a rapid convergence rate with small sample autocorrelations for the samples in our applications.

3.1. *Model selection*

Most existing literature focuses on estimating and selecting copula functions, conditioned on estimated marginal distributions. Here, we summarize alternative model evaluation measures for copula-based distributions. Let $L(\mathbf{x}|\Psi_k, \mathcal{M}_k)$ represent the likelihood function for model $\mathcal{M}_k$, including marginal distributions and the copula function. The deviance function is $D(\Psi_k) = -2 \log L(\mathbf{x}|\Psi_k, \mathcal{M}_k)$.

Common model selection criteria include the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), their expected versions (EAIC and EBIC), and the Deviance Information Criterion (DIC), defined as:

$$AIC(\mathcal{M}_k) = D(E[\Psi_k|\mathbf{x}, \mathcal{M}_k]) + 2d_k, \quad BIC(\mathcal{M}_k) = D(E[\Psi_k|\mathbf{x}, \mathcal{M}_k]) + \log(n)d_k,$$

$$EAIC(\mathcal{M}_k) = E[D(\Psi_k)|\mathbf{x}, \mathcal{M}_k] + 2d_k, \quad EBIC(\mathcal{M}_k) = E[D(\Psi_k)|\mathbf{x}, \mathcal{M}_k] + \log(n)d_k,$$

$$DIC(\mathcal{M}_k) = 2E[D(\Psi_k)|\mathbf{x}, \mathcal{M}_k] - D(E[\Psi_k|\mathbf{x}, \mathcal{M}_k]).$$

Further details on all these measures can be found in Spiegelhalter, Best, Carlin and Linde [23].

For a posterior sample $\{\Psi_k^{(1)}, \ldots, \Psi_k^{(L)}\}$, the Monte Carlo approximations to $E[D(\Psi_k)|\mathbf{x}, \mathcal{M}_k]$ and $E[\Psi_k|\mathbf{x}, \mathcal{M}_k]$ are:

$$L^{-1} \sum_{\ell=1}^{L} D(\Psi_k^{(\ell)}) \quad \text{and} \quad L^{-1} \sum_{\ell=1}^{L} \Psi_k^{(\ell)}.$$

These measures are invariant under monotone transformations of the marginal distributions, which is useful in copula modeling. These criteria, computationally efficient, account for parameter dependence and can be applied to any copula family with computable densities.

4. Applications

Our main goal in this section is to demonstrate the applicability of the continuous mixture in copulas to real data sets. Two other important features are the Bayesian approach and model selection via DIC (and similar criteria). Actuarial data on insurance claim requirements [7] are fitted to the mixture model. Additionally, two financial data sets are analyzed: the daily market indices of Amsterdam and Frankfurt, and those of Mexico and Argentina [14, 19].

Due to stylized facts in finance, we are interested in modeling with asymmetric distributions and heavy tails. For this, we use the approach of Fernández and Steel [4]. In particular, to illustrate the main idea of the continuous mixture, we have the asymmetric normal distribution given by:

$$f_i(x_i|\mu_i, y_i, \gamma_i) = \frac{2\gamma_i}{\gamma_i^2+1}\left[\phi\left(\frac{x_i-\mu_i}{\gamma_i\sqrt{y_i}}\right)I_{[0,\infty)} + \phi\left(\frac{\gamma_i(x_i-\mu_i)}{\sqrt{y_i}}\right)I_{(-\infty,0)}\right]$$

which is used as the marginal distributions of $C_{\boldsymbol{X}|\boldsymbol{Y}}$ in all 3 applications, with $C_{\boldsymbol{X}|\boldsymbol{Y}}$ being one of the six families of copulas from Table 1. Here, ϕ corresponds to the pdf of a standard normal distribution, and given Y_i, y_i represents the scale of the distribution. Additionally, Y_i has an inverse gamma distribution with parameters $\eta_i/2$ and $\eta_i\lambda_i^2/2$. However, we take $C_{\boldsymbol{Y}}$ simply as the product copula. Notice that X_i has an asymmetric Student-t distribution. This model is capable of capturing simpler structures in the data with its particular cases. Other choices could be made for the marginal distributions and copulas. For all applications, we generated 11,000 iterations of the Gibbs algorithm, discarded the first 1,000 iterations, and took 1 value every 10 samples, resulting in a sample of size 1,000 from the posterior distributions of each model. The degrees of freedom of the Student copula was fixed at 4 for all applications.

4.1. *Insurance claim requirements*

Figure 1 shows the data set corresponding to insurance claim requirements with 1,500 observations of adjusted loss expenses (ALAE) and claim payments (*Loss*), both on the logarithmic scale. The statistical summary of these two measures shows that both appear to be left-skewed, with a correlation coefficient of 0.425 and the Kendall's τ measure equal to 0.302. A better description of the data can be found in Frees and Valdez [7].

Table 2 presents the DIC and the other criteria for all the copula mixture models. For the AIC, EAIC, BIC, and EBIC, we fixed the number of parameters at 11, corresponding to 3 for each asymmetric normal marginal, 2 for each inverse gamma, and 1 for the copula. The Gumbel copula provides the best fit compared to the others, although it had values close to the heavy-tail copula.

In Table 3, we have the summary of the posterior distribution for the Gumbel copula model. Note that both variables have a high probability of being left-skewed, as indicated by the credibility interval and the mean being below 1. Additionally,

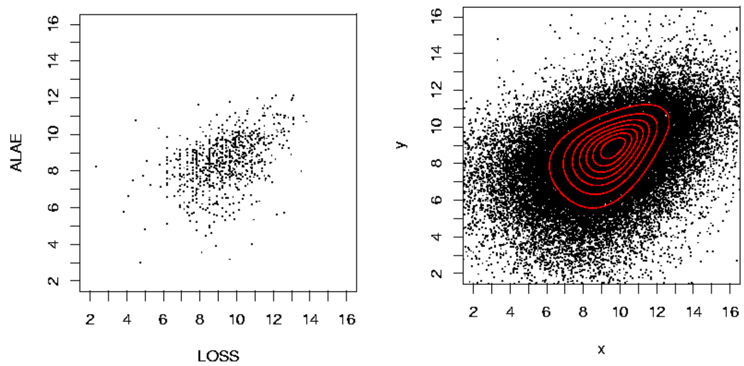

Figure 1. Insurance claim requirements (Loss e ALAE). Real data on the left and simulated data on the right.

Table 2. DIC, pd, AIC, EAIC, BIC, and EBIC for the six copula models for the loss request data of the insurance company.

Copula	DIC	pd	AIC	EAIC	BIC	EBIC
Clayton	5,375	6.77	5,383	5,390	5,442	5,449
Frank	5,175	7.07	5,183	5,190	5,242	5,249
Gaussian	5,173	6.78	5,182	5,189	5,240	5,247
Gumbel	**5,127**	7.01	**5,135**	**5,142**	**5,194**	**5,201**
Heavy tail	5,139	7.08	5,147	5,154	5,205	5,212
Student	5,189	6.85	5,198	5,205	5,256	5,263

both have heavy tails. The copula parameter, θ, is concentrated around 1.4, indicating positive dependence between the variables. Figure 1 shows 50,000 pairs of values from the Gumbel copula model and the contour using the Rao-Blackwell estimator [8]. The same sample was used to estimate a Kendall's τ equal to 0.290. The overall model captured important characteristics of the data, such as the dependence between the variables, the asymmetries, and heavy tails in both marginals.

Table 3. Summary of the posterior distribution of the Gumbel copula model based on a sample of size 1000. SD, 2.5%, and 97.5% are the standard deviation and percentiles of the posterior distribution of the parameters, respectively.

Ψ	Loss					ALAE				
	Mean	SD	2,5%	Median	97,5%	Mean	SD	2,5%	Median	97,5%
μ	9.481	0.099	9.316	9.486	9.644	8.895	0.076	8.772	8.893	9.026
λ^2	2.504	1.516	0.407	2.279	5.397	1.810	1.187	0.325	1.644	3.943
γ	0.957	0.033	0.907	0.955	1.012	0.840	0.026	0.797	0.841	0.883
η	9.276	10.37	2.150	5.098	32.44	7.873	8.955	2.147	4.623	26.07
θ	1.442	0.033	1.387	1.441	1.498	-	-	-	-	-

4.2. *Amsterdam and Frankfurt indices*

The data correspond to 2,733 daily returns from the Amsterdam and Frankfurt indices, from July 10, 1987, to December 31, 1997 [6]. We are primarily interested in studying the dependence structure, so we first fit GARCH(1,1) models with normal errors for both indices and apply the continuous mixture model to the residuals. Statistical tests showed that all estimated parameters were statistically significant. Figure 2 shows the dispersion of the residuals from the GARCH(1,1) models for the Amsterdam and Frankfurt returns.

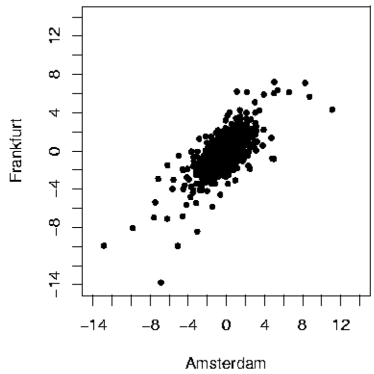

Figure 2. Residuals from the GARCH(1,1) fit for both the Amsterdam and Frankfurt indices.

The Table 4 shows the DIC and other criteria. For the AIC, EAIC, BIC, and EBIC, we set the number of parameters to 11, corresponding to 3 for each asymmetric normal marginal, 2 for each inverse gamma, and 1 for the copula. The Student copula provides a better fit than the others, although it yields values close to the Gaussian copula. It is worth noting that the degrees of freedom of the Student copula are fixed at 4.

Table 4. DIC, pd, AIC, EAIC, BIC, and EBIC for the six copula models for the GARCH(1,1) residuals of the Amsterdam and Frankfurt indices.

Copula	DIC	pd	AIC	EAIC	BIC	EBIC
Clayton	6,284	6.90	6,293	6,300	6,358	6,365
Frank	5,669	6.94	5,677	5,684	5,742	5,749
Gaussian	5,533	6.48	5,542	5,548	5,607	5,613
Gumbel	5,710	6.85	5,719	5,726	5,784	5,791
Heavy tail	6,376	6.89	6,384	6,391	6,449	6,456
Student	**5,517**	6.55	**5,526**	**5,533**	**5,591**	**5,598**

In Table 5, we present the summary of the posterior distribution for the Student copula model. Note that both variables have a high probability of being left-skewed, as indicated by the credibility interval and mean below 1; moreover, both have heavy tails. The copula parameter, θ, is concentrated around 0.7, indicating positive dependence between the two markets. The overall model captured important characteristics of the data, such as the dependence between them, the skewness, and the heavy tails in both marginals.

Table 5. Summary of the posterior distribution of the Student copula model based on a sample of size 1,000. SD, 2.5%, and 97.5% are the standard deviation and percentiles of the posterior distribution of the parameters, respectively.

Ψ	Amsterdam					Frankfurt				
	Mean	SD	2,5%	Median	97,5%	Mean	SD	2,5%	Median	97,5%
μ	0.126	0.034	0.068	0.125	0.182	0.103	0.040	0.038	0.104	0.168
λ^2	1.309	0.874	0.208	1.149	3.060	1.591	1.067	0.280	1.402	3.540
γ	0.899	0.016	0.872	0.900	0.924	0.918	0.016	0.893	0.917	0.945
η	8.533	9.898	2.128	4.640	31.55	8.260	9.109	2.164	4.510	28.58
θ	0.708	0.013	0.686	0.709	0.728	-	-	-	-	-

4.3. *Indices of Mexico and Argentina.*

In this application, we aim to show that data of the same nature can have different dependency structures. The data corresponds to 3,132 daily returns of the indices from Mexico and Argentina from December 22, 1992, to December 22, 2004 [1]. We are primarily interested in studying the dependency structure, so we first fit GARCH(1,1) models with normal errors for both indices and apply the mixture model to the residuals. All the estimated parameters were statistically significant. Figure 3 shows the dispersion of the residuals from the GARCH(1,1) models for the returns of Mexico and Argentina.

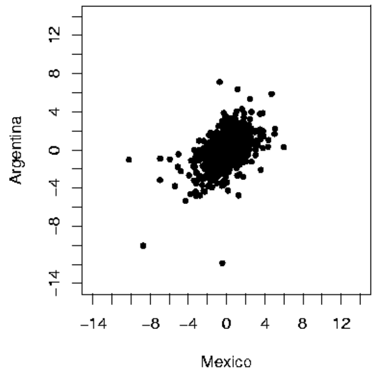

Figure 3. Residuals from the GARCH(1,1) fit for both the Mexico and Argentina.

Table 6 shows the DIC and other criteria. For the AIC, EAIC, BIC, and EBIC, we fixed the number of parameters at 11, corresponding to 3 for each asymmetric normal marginal, 2 for each inverse gamma, and 1 for the copula. The Frank copula model has a better fit than the others, although it showed values close to the Gaussian.

Table 6. DIC, pd, AIC, EAIC, BIC, and EBIC for the six copula models for the GARCH(1,1) residuals of the Mexico and Argentina indices.

Copula	DIC	pd	AIC	EAIC	BIC	EBIC
Clayton	5,792	7.03	5,800	5,807	5,866	5,873
Frank	**5,501**	7.18	**5,509**	**5,516**	**5,576**	**5,583**
Gaussian	5,538	6.96	5,547	5,554	5,613	5,620
Gumbel	5,685	7.07	5,692	5,699	5,759	5,766
Heavy tail	5,930	6.98	5,938	5,945	6,004	6,011
Student	5,586	7.09	5,594	5,601	5,660	5,668

In Table 7, we have the summary of the posterior distribution for the Frank copula model. Some characteristics given by the estimation are positions away from $(0, 0)$, scale parameters near 1, asymmetries, and heavy tails for the marginals. Using ar-

Table 7. Summary of the posterior distribution of the Frank copula model based on a sample of size 1,000. SD, 2.5%, and 97.5% are the standard deviation and percentiles of the posterior distribution of the parameters, respectively.

Ψ	Mexico					Argentina				
	Mean	SD	2,5%	Median	97,5%	Mean	SD	2,5%	Median	97,5%
μ	0.104	0.032	0.050	0.105	0.158	0.074	0.029	0.028	0.076	0.121
λ^2	1.089	0.713	0.238	0.965	2.358	1.038	0.636	0.218	0.930	2.252
γ	0.917	0.017	0.888	0.916	0.945	0.935	0.016	0.910	0.935	0.961
η	7.782	8.513	2.165	4.526	25.28	8.605	9.967	2.142	4.661	31.56
θ	4.220	0.152	3.967	4.223	4.465	-	-	-	-	-

tificial data from the Frank copula model and taking the posterior mean of the parameters, we have an estimated Kendall's τ of 0.407. The model captured important features of the data such as dependence, asymmetries, and heavy tails in both marginals.

5. Concluding remarks

In this article, we introduce a method for constructing new families of copulas by combining existing ones. This approach involves copula mixtures based on their marginal distributions, extending prior work in the literature. We focus particularly on continuous mixtures, specifically through a scale mixture of an asymmetric normal distribution and an inverse gamma distribution. We review various measures of dependence and demonstrate how these can be calculated within the mixture model. Monte Carlo integration is employed for all measures, including non-parametric sample versions of Kendall's τ and Spearman's ρ. Inference in the copula mixture model is performed using the Bayesian framework with Markov chain Monte Carlo (MCMC) techniques. Model selection is based on the deviance information criterion (DIC) and similar criteria. To illustrate the methodology, we present an example using continuous random variables. Additionally, real-world datasets are used to demonstrate the practical applicability of the copula mixture model.

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Appendix A. Copula densities

Table A1. Six copula densities. $\phi_{(X,Y)}(.)$, $\phi(.)$, $\Phi(.)$, $t_{(X,Y),\nu}(.)$, $t_\nu(.)$ and $T_\nu(.)$ are the bivariate pdf with correlation coefficient θ, the univariate pdf, and the cdf of standard normal random variables, and standard Student-t random variables, respectively.

Cópula	$c(u,v\mid\theta) = \frac{\partial^2 C(u,v\mid\theta)}{\partial u \partial v}$
Clayton	$(1+\theta)(uv)^{-(\theta+1)}(u^{-\theta}+v^{-\theta}-1)^{-(1/\theta+2)}$
Frank	$\dfrac{-\theta e^{-\theta(u+v)}(e^{-\theta}-1)}{\left[e^{-\theta(u+v)}-e^{-\theta u}-e^{-\theta v}+e^{-\theta}\right]^2}$
Gaussian	$\dfrac{\phi_{(X,Y)}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi_X(\Phi^{-1}(u))\phi_Y(\Phi^{-1}(v))}$
Gumbel	$(uv)^{-1}(-ln(u))^{\theta-1}(-ln(v))^{\theta-1}\left(\theta-1+[(-\log u)^\theta+(-\log v)^\theta]^{1/\theta}\right)$ $\times[(-\log u)^\theta+(-\log v)^\theta]^{1/\theta-2}\exp\left\{-[(-\log u)^\theta+(-\log v)^\theta]^{1/\theta}\right\}$
Heavy tail	$(1+1/\theta)[(1-u)(1-v)]^{-(1/\theta+1)}[(1-u)^{-1/\theta}+(1-v)^{-1/\theta}-1]^{-(\theta+2)}$
Student	$\dfrac{t_{(X,Y),\nu}(T_\nu^{-1}(u),T_\nu^{-1}(v))}{t_{X,\nu}(T_\nu^{-1}(u))t_{Y,\nu}(T_\nu^{-1}(v))}$